Time in quantum mechanics

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Time is often said to play an essentially different role from position in quantum mechanics: whereas position is represented by a Hermitian operator, time is represented by a c-number. This difference is puzzling and has given rise to a vast literature and many efforts at a solution. It is argued that the problem is only apparent and that there is nothing in the formalism of quantum mechanics that forces us to treat position and time differently. The apparent problem is caused by the dominant role point particles play in physics and can be traced back to classical mechanics.
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I. INTRODUCTION

From the early days, the role time plays in quantum mechanics has caused concerns. For example, von Neumann, in his famous book complains: "First of all we must admit that this objection points at an essential weakness which is, in fact, the chief weakness of quantum mechanics: its nonrelativistic character, which distinguishes the time t from the three space coordinates x, y, z, and presupposes an objective simultaneity concept. In fact, while all other quantities (especially those x, y, z closely connected with t by the Lorentz transformation) are represented by operators, there corresponds to the time an ordinary number-parameter t, just as in classical mechanics."[1] Of course, it is true that elementary quantum mechanics is not relativistic, but it is not true that the three space coordinates are operators in quantum mechanics. Seventy years later little seems to have changed when we read: "Moreover, space and time are treated very differently in quantum mechanics. The spatial coordinates are operators, whereas time is a parameter."[2]

Most textbooks written during the intervening period tell us that time is exceptional in quantum mechanics and many efforts to deal with this problem have appeared in the literature.[3] In the following, I will show that time does not pose a special problem for quantum mechanics.[4]

II. TIME IN CLASSICAL MECHANICS

Quantum mechanics is modeled on classical Hamiltonian mechanics. In Hamiltonian mechanics a physical system is described by N pairs of canonical conjugate dynamical variables, Q_k and Π_k, which satisfy the following Poisson-bracket relations:[5]

$$\{Q_k,\Pi_l\}=\delta_{kl}, \quad \{Q_k,Q_l\}=\{\Pi_k,\Pi_l\}=0. \tag{1}$$

The canonical variables define a point in the $2N$-dimensional phase space of the system. The time evolution of the system is generated by the Hamiltonian, a function of the canonical variables, $H=H(Q_k,\Pi_k)$,

$$dQ_k/dt=\{Q_k,H\}, \quad d\Pi_k/dt=\{\Pi_k,H\}. \tag{2}$$

(We assume that H does not explicitly depend on time.)

The Q_k and Π_k are generalized variables; they need not be positions and momenta, but may be angles, angular momenta, and the like. However, if the system is a collection of point particles, the canonical variables are usually taken to be the positions $\boldsymbol{q}_n$ and momenta $\boldsymbol{p}_n$ of the particles (three-vectors are in bold type and the subscript denotes the nth particle).

Let us consider the relation of the Hamiltonian formalism with space and time. In all of physics, with the exception of Einstein's theory of gravity (general relativity), physical systems are assumed to be situated in a three-dimensional Euclidean space. The points of this space are given by Cartesian coordinates $\boldsymbol{x}=(x,y,z)$. Together with the time parameter t, they form the coordinates of a continuous, independently given, space-time background. How the existence of this space and time background is to be justified is an important and difficult question into which I will not enter. I will just take this assumption as belonging to the standard formulation of classical and quantum mechanics and of special relativity.

The $(3+1)$-dimensional space-time must be sharply distinguished from the $2N$-dimensional phase space of the system, and the space-time coordinates $(\boldsymbol{x},t)$ must be sharply distinguished from the dynamical variables (Q_k,Π_k) characterizing physical systems *in* space-time. In particular, the position variable $\boldsymbol{q}$ of a point particle must be distinguished from the coordinate $\boldsymbol{x}$ of the space-point the particle occupies, although we have the numerical relation: $q_x=x,\ q_y=y,\ q_z=z$. A point particle is a material system having a mass, a position, a velocity, an acceleration, while $\boldsymbol{x}$ is the coordinate of a fixed point of space. We will see that mixing up $\boldsymbol{q}$ and $\boldsymbol{x}$ is at the root of the problem of *time* in quantum mechanics.

A vital role is played in physics by the symmetries that space and time are supposed to possess. It is assumed that three-dimensional space is isotropic (rotationally symmetric) and homogeneous (translationally symmetric) and that there is translational symmetry in time. In special relativity the space-time symmetry is extended by Lorentz transformations which mix $\boldsymbol{x}$ and t, transforming them as the components of a four-vector. (In a relativistic context I shall write spacetime instead of space-time.)

In the following it is important to note that individual physical systems *in* space-time need not show these symmetries; only the physical laws, that is, the totality of physically allowed situations and processes, must show them. A given physical system need not be rotationally invariant, and a position variable of a physical system need not be part of a four-vector.

The generators of translations in space and time are the total momentum $\boldsymbol{P}$ and the total energy H, respectively.[6] The generator of space rotations is the total angular momentum $\boldsymbol{J}$. We shall in particular be interested in the behavior of dynamical variables under translations in time and space. For an infinitesimal translation $\delta\tau$ in time, we have

$$\delta Q_k = \{Q_k, H\}\delta\tau, \quad \delta\Pi_k = \{\Pi_k, H\}\delta\tau, \tag{3}$$

and for an infinitesimal translation $\delta\boldsymbol{a}$ in space

$$\delta Q_k = \{Q_k, \boldsymbol{P}\}\cdot\delta\boldsymbol{a}, \quad \delta\Pi_k = \{\Pi_k, \boldsymbol{P}\}\cdot\delta\boldsymbol{a}. \tag{4}$$

At this point one may wonder why the Hamiltonian, the generator of time translations, that is, of the time evolution of the system, is so much more prominent in classical mechanics than is the total momentum, the generator of translations in space. The reason is that the dynamical variables of the systems that are traditionally studied in classical mechanics, namely particles and rigid bodies, transform trivially under space translations. For example, for a system of particles, a space translation $\boldsymbol{a}$,

$$\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}, \quad t \to t, \tag{5}$$

induces the simple transformation

$$\boldsymbol{q}_n \to \boldsymbol{q}_n + \boldsymbol{a}, \quad \boldsymbol{p}_n \to \boldsymbol{p}_n, \tag{6}$$

of the canonical variables. The infinitesimal form of transformation (6) is $\delta\boldsymbol{q}_n = \delta\boldsymbol{a}$, $\delta\boldsymbol{p}_n = 0$, and by comparing this form with Eq. (4), we find the simple relations

$$\{q_{n,i}, P_j\} = \delta_{ij}, \quad \{p_{n,i}, P_j\} = 0 \quad (i,j = x,y,z) \tag{7}$$

with the obvious solution $\boldsymbol{P} = \Sigma\boldsymbol{p}_n$. If there is only one particle and one space dimension, Eq. (7) becomes

$$\{q, P\} = 1, \quad \{p, P\} = 0. \tag{8}$$

The simplicity of behavior under space translations shown by a point particle is not a general feature of the theory. If the physical system is a *field* such as, for example, an electromagnetic field strength or the density distribution in a fluid, the effect of a translation in space may be as nontrivial as the effect of a translation in time and H and $\boldsymbol{P}$ are equally important. In relativity theory, H and $\boldsymbol{P}$ are combined into the components of a four-vector.

The great similarity between the behavior under space translations (and rotations) of the position $\boldsymbol{q}$ of a point particle and the coordinate $\boldsymbol{x}$ of a point of three-dimensional space obscures the conceptual difference between the two, and the widespread use of the notation $\boldsymbol{x}$ for the position of a particle has greatly added to this confusion. In many discussions in classical mechanics, an explicit distinction between $\boldsymbol{x}$ and $\boldsymbol{q}$ is never made. Although ignoring the distinction may be innocent when it is sufficiently clear what is meant, it has, nevertheless, caused important misunderstandings of a general kind of which we will see examples here and in Sec. III.

In relativity theory the coordinates $\boldsymbol{x}, t$ transform as the components of a Lorentz four-vector. This has led many to believe that the position $\boldsymbol{q}$ of a particle should also be part of a four-vector with the time coordinate t as the fourth component. But $\boldsymbol{q}$ is a dynamical variable belonging to a material system, whereas t is a universal spacetime coordinate. No one would think of adding t to the position variables of an arbitrary physical system, say a rigid body, to form a four-vector. It is only the great resemblance of a point particle to a space point that has misled people in this case. Already the case of a system consisting of several particles should be eye-opening. In this case we would have to combine the very same t with *all* position variables.

As remarked above, a spacetime symmetry does not imply the same symmetry of every physical system in spacetime. The point particle is a case in point. It simply does not possess a dynamical variable that may be combined with its position variable to form a four-vector. The position of a point particle is an essentially noncovariant concept.[7] (On the other hand, its momentum and energy do form a four-vector.)

Another confusion, possibly related to the above one, lies at the root of efforts to include the time parameter t in the set of canonical variables as the partner conjugate to H. Again, because H and t differ conceptually in the same way as do $\boldsymbol{q}$ and t, such efforts are misconceived. In fact, H, being a given function of the original canonical variables, is not an independent canonical variable. Such an effort, therefore, implies a severe departure from the original scheme. Had the roles of $\boldsymbol{x}$ and $\boldsymbol{P}$, the analogs of t and H, been more clearly recognized in classical mechanics, the temptation to add t to the canonical variables, while leaving $\boldsymbol{x}$ alone, would probably not have arisen.

Time as a dynamical variable. The above discussion served to emphasize the conceptual difference between the space-time coordinates and the dynamical variables of physical systems in space-time. In particular, the universal time coordinate t should not be mixed with dynamical position variables. But do physical systems exist that have a dynamical variable that resembles the time coordinate t in the same way as the position variable $\boldsymbol{q}$ of a point particle resembles the space coordinate $\boldsymbol{x}$? The answer is yes! Such systems are *clocks*. A clock stands, ideally, in the same simple relation to the universal time coordinate t as a point particle stands to the universal space coordinate $\boldsymbol{x}$. We may generally define an ideal clock as a physical system describable by a dynamical variable that, under time translations, behaves similarly to the time coordinate t. Such a variable, which we shall call a clock variable, or more generally, a time variable, may be a pointer position or an angle or even a momentum. Just as a position variable indicates the position of a system in space, a clock variable indicates the position of a system in time. The closest analog to the transformations (5) and (6) would be a linear clock variable θ, with a conjugate momentum η, such that a time translation

$$\boldsymbol{x} \to \boldsymbol{x}, \quad t \to t + \tau, \tag{9}$$

induces the simple transformation

$$\eta \to \eta, \quad \theta \to \theta + \tau. \tag{10}$$

If we compare the infinitesimal form of this transformation with Eq. (3), we find

$$\{\eta, H\} = 0, \quad \{\theta, H\} = 1, \tag{11}$$

which is the analog of Eq. (8) for time variables.

The simplest solution of Eq. (11) is $H(\theta,\eta) = \eta$. This solution is analogous to the case of a single particle where the total momentum $P(q,p)$ coincides with the momentum p. The equation of motion, $d\theta/dt = \{\theta, H\} = 1$, has the solution $\theta = t$ which is the analog of the relation $q = x$ for a point particle.

A model of a linear clock is provided by a particle moving in a constant force field.[8] The momentum of the particle is a linear function of t and furnishes a time variable. More precisely, starting from the Hamiltonian $H(q,p)=p^2/2m-Kq$ of the particle in the field K, we go over to the variables (θ,η) by the canonical transformation:

$$q\rightarrow\theta=p/K,\ p\rightarrow\eta=H=(p^2/2m-Kq). \tag{12}$$

Then $\{q,p\}=\{\theta,\eta\}=1$ and $H(\theta,\eta)=\eta$. It follows that $d\theta/dt=1$. Note that H is unbounded and may take on any real value.

Actual clocks are not ideal in the sense of Eq. (11); in fact, most real clocks, such as, for example, a pendulum clock or a quartz clock, are not even continuous indicators of time. What the example purports to show is that the Hamiltonian formalism allows for the existence of systems satisfying Eq. (11) that play the same role with respect to H and t as point particles do with respect to P and x.

The cyclic clock variable corresponding to our linear clock is an angle variable ϕ with conjugate momentum L and Hamiltonian $H(\phi,L)=L$. Here $\phi=t(\bmod 2\pi)$. We shall come back to these examples in Sec. IV.

We conclude that in classical physics a sharp distinction must be made between the universal space-time coordinates and the dynamical variables of specific physical systems situated *in* space-time. Particles and clocks are physical systems having dynamical variables which behave in much the same way as do the space and time coordinates, respectively, and may thus serve to indicate the position of the system in space and in time. Point particles and clocks are basically noncovariant concepts. If one is to look for physical systems that transform covariantly under relativistic spacetime transformations, one must consider *fields*.

III. TIME IN QUANTUM MECHANICS

In quantum mechanics the situation is not essentially different. The theory assumes a fixed, unquantized space-time background, the points of which are given by c-number coordinates $\boldsymbol{x},t$. The space-time symmetry transformations are expressed in terms of these coordinates.

Dynamical variables of physical systems, on the other hand, are quantized: they are replaced by operators on a Hilbert space.[9] All formulas of the preceding section remain valid if the Poisson brackets are replaced by commutators according to $\{,\}\rightarrow(i\hbar)^{-1}[,]$. In particular, the canonical variables are replaced by operators that satisfy the commutation relations:[10]

$$[Q_k,\Pi_l]=i\hbar\,\delta_{kl}, \quad [Q_k,Q_l]=[\Pi_k,\Pi_l]=0. \tag{13}$$

(In this section, symbols representing dynamical variables are assumed to be operators.) Thus, for a point particle,

$$[q_i,p_j]=i\hbar\,\delta_{ij}, \quad [q_i,q_j]=[p_i,p_j]=0, \tag{14}$$

where $(i,j=x,y,z)$ denotes the Cartesian components of the position $\boldsymbol{q}$ and momentum $\boldsymbol{p}$ of the particle. These relations have the well-known representation where $\boldsymbol{q}$ is the multiplication operator and $\boldsymbol{p}$ the corresponding differential operator. Both of these operators are unbounded and have the complete real axis as their spectrum. However, if the position wave functions are required to obey periodic boundary conditions, the eigenvalues of $\boldsymbol{p}$ become discrete, and if the position wave functions are required to vanish at the endpoints of a finite interval (particle in a box), a self-adjoint momentum operator does not even exist.

Corresponding statements hold for $\boldsymbol{q}$. Similarly, because the wave functions of an angle variable must obey a periodic boundary condition, the eigenvalues of the corresponding angular momentum operator are discrete. Discrete energy eigenvalues are of course the hallmark of quantum mechanics. Nobody would conclude from these facts that something is wrong with the notions of position, momentum, angular momentum, or energy in quantum mechanics. One should keep this typical quantum mechanical behavior in mind when we discuss quantum mechanical time operators.

But first, let me point out some of the confusion that has established itself in standard presentations of quantum mechanics as a result of mixing up $\boldsymbol{q}$ and $\boldsymbol{x}$. Most elementary quantum mechanics texts start by considering a single point particle. The particle position is commonly denoted by $\boldsymbol{x}$ (instead of by $\boldsymbol{q}$) and the time-dependent wave function is written as $\psi(\boldsymbol{x},t)$. This notation is misleading in several ways. It gives the false impression that the wave function is just an ordinary wave in three-dimensional space, an impression that is reinforced by the usual discussions of double slit interference and quantum tunnelling. It seems that even von Neumann has fallen victim to this notation.[1] However, contrary to an ordinary field, such as for example, the electromagnetic field, ψ is a highly abstract entity, living in an abstract configuration space, carrying no energy and momentum but only *information* about the results of measurements. Furthermore, the notation suggests that $\boldsymbol{x}$ and t are quantities of the same type and leads to the question why t, the universal time coordinate, is not an operator like $\boldsymbol{x}$. The notation $\psi(\boldsymbol{q},t)$ for the wave function, where $\boldsymbol{q}$ denotes an eigenvalue of the position operator, would certainly have made this question a less obvious one to ask.[11]

Again, the idea, criticized in Sec. II, that t can be interpreted as the canonical variable conjugate to the Hamiltonian, might lead one to expect t to obey the canonical commutation relation $[t,H]=i\hbar$. But if t is the universal time operator, it should have continuous eigenvalues running from $-\infty$ to $+\infty$, and from this property, the same would follow for the eigenvalues of any H. But we know that discrete eigenvalues of H may occur. From this reasoning Pauli concluded that the introduction of an operator t is basically forbidden and the time must necessarily be considered as an ordinary number (c-number).[12] In this way the unsolvable problem of time in quantum mechanics has arisen.

Note that it is crucial for this argument that t is supposed to be a universal operator, valid for all systems. According to Pauli, the introduction of such an operator is basically forbidden because *some* systems have discrete energy eigenvalues.

From our previous discussion it should be clear that the universal time coordinate t is the partner of the space coordinates $\boldsymbol{x}$. Neither the space coordinates nor the time coordinate is quantized in standard quantum mechanics. So, the above problem simply does not exist. If we are to look for a time *operator* in quantum mechanics, we should not try to quantize the universal time coordinate but consider timelike (in the literal sense) dynamical variables of specific physical systems, that is, clocks. Because a clock variable is an ordinary dynamical variable, quantization should not, in principle, be especially problematic. However, we must be prepared to encounter the well-known quantum effects mentioned above: a dynamical system may have a continuous time variable, or a discrete one, or no time variable at all. But the discreteness, say, of the eigenvalues of a time operator invalidates the notion of time in quantum mechanics as little as does the discreteness of the eigenvalues of an energy operator invalidates the notion of energy. Let us now turn to the quantum version of the clocks considered in Sec. II.

IV. QUANTUM CLOCKS

In Sec. II we characterized the ideal time variables by their behavior under time translations, that is, by relations (11). The analogous relations for the corresponding quantum mechanical operators are

$$[\eta,H]=0, \quad [\theta,H]=i\hbar. \tag{15}$$

A. The linear quantum clock

The equations in (15) are satisfied by the quantum version of our simple linear clock if we take θ to be the multiplication operator, $\eta=-i\hbar d/d\theta$ and $H=\eta$. The operators θ and η satisfy $[\theta,\eta]=i\hbar$. The spectrum of θ, η, and H is the entire real axis.

It is illuminating to compare this result with Pauli's argument above. Our θ resembles the universal time parameter t as much as possible, and our H has indeed continuous eigenvalues. But this result does not imply that the Hamiltonians of *other* physical systems also must have continuous eigenvalues. It only means that in such systems time operators either do not exist or cannot resemble t as closely as does our linear clock (although they can come pretty close, as we shall see). It is the supposed universality of the time operator that is crucial for the validity of Pauli's argument.

But another objection must be mentioned. Most physically interesting systems have Hamiltonians that are bounded from below. In such systems a time operator having the real axis as its spectrum does not exist. It is sometimes asserted that the Hamiltonian of real physical systems *must* be bounded from below in order to guarantee the stability of the system. The archetypical example is the Rutherford atom whose very instability gave rise to Bohr's atomic theory and to the development of modern quantum mechanics. However, the instability of the Rutherford atom is caused by its interaction with the electromagnetic field, which allows the atom to dissipate its energy in the form of electromagnetic radiation. Without this interaction, the atom would be stable because of conservation of energy. Similarly, in quantum mechanics, the stationary states of the hydrogen atom are stationary as long as the interaction of the atom with the electromagnetic field is not taken into account. So, for an isolated system, the demand that its Hamiltonian has a lower bound is not at all necessary. Thus, from the point of view of the quantum mechanical formalism, our linear clock is a *bona fide* physical system.

It is ironic that the demand that H be bounded from below precludes the existence of an acceptable particle *position* operator in relativistic quantum mechanics (see Sec. V).

B. The continuous cyclic quantum clock

This clock is characterized by an angle variable ϕ that will play the role of time variable. In quantum mechanics ϕ is represented by an operator Φ. The Hilbert space in the angle representation consists of the square-integrable functions f of ϕ on the interval $[0,2\pi]$. The operators of angle and angular momentum are represented by ($\hbar=1$),

$$\Phi f(\phi)=\phi f(\phi), \tag{16}$$

$$L f(\phi)=-i\, df(\phi)/d\phi. \tag{17}$$

The operator Φ is self-adjoint on the entire Hilbert space, whereas L is self-adjoint on the subspace of the square integrable, differentiable functions satisfying $f(0)=f(2\pi)$. These operators have complete, orthonormal sets of generalized eigenstates $|\phi\rangle$ and $|m\rangle$:

$$\Phi|\phi\rangle=\phi|\phi\rangle, \quad \langle\phi|\phi'\rangle=\delta(\phi-\phi'), \tag{18}$$

$$L|m\rangle=m|m\rangle, \quad \langle m|m'\rangle=\delta_{m,m'}, \tag{19}$$

where the eigenvalue ϕ runs through the interval $[0,2\pi]$ and $m=0,\pm 1,\pm 2$.

In the ϕ representation the states $|\phi\rangle$ and $|m\rangle$ are represented by the wave functions $\langle\phi|\phi'\rangle=\delta(\phi-\phi')$ and $\langle\phi|m\rangle=(2\pi)^{-1/2}e^{im\phi}$, respectively. The situation is very similar to that of the linear clock, except for the fact that the interval on which the functions $f(\phi)$ are defined is finite. In particular, we have

$$|\phi\rangle=(2\pi)^{-1/2}\sum_{-\infty}^{+\infty} e^{-im\phi}|m\rangle, \tag{20}$$

where the sum runs over all values of m.

The dynamics of the system is introduced by specifying a Hamiltonian; we let $H=\omega L$, where ω is a constant frequency. The operator of translations in time is $U(t)=e^{-iHt}$, and with the help of Eq. (20) we find

$$U(t)|\phi\rangle=e^{-iHt}|\phi\rangle=(2\pi)^{-1/2}\sum e^{-im\phi-im\omega t}|m\rangle$$
$$=|\phi+\omega t\rangle, \tag{21}$$

which is precisely the behavior we expect of the hand of a clock: it rotates at constant angular velocity and after an arbitrarily short time, an eigenstate $|\phi\rangle$ of the hand position goes to an orthogonal state. Letting $\omega=1$, we see that ϕ plays the role of a time variable: under a time translation it behaves exactly as t does. The energy of this continuous clock is unbounded from below and above, but the energy values are discrete. Returning once more to Pauli's argument, we see that discrete energy eigenvalues do not rule out the existence of a well-behaved time operator, even though its spectrum is not the real axis.

Let us now see what happens if we restrict the energy of our clock.

C. The discrete cyclic quantum clock (Ref. 13)

Let us restrict the sum in Eq. (20) to values of m satisfying the condition $-l\leqslant m\leqslant l$, where l is a positive integer, and consider the $2l+1$ orthogonal states

$$|\phi_k\rangle=(2l+1)^{-1/2}\sum_{m=-l}^{l} e^{-im\phi_k}|m\rangle, \tag{22}$$

where ϕ_k takes the values

$$\phi_k=2\pi k/(2l+1),\ k=-l,\ldots,l. \tag{23}$$

We may now define a time operator

$$\Theta=\sum_{k=-l}^{l}\phi_k|\phi_k\rangle\langle\phi_k|. \tag{24}$$

The eigenvalues of Θ are the $(2l+1)$ discrete times ϕ_k, and if the system is in the eigenstate $|\phi_k\rangle$ at time t, it will be in the eigenstate $|\phi_{k+1}\rangle$ at time $t+2\pi/(2l+1)$, as may easily be verified by applying the evolution operator $U(t)$ to $|\phi_k\rangle$. This behavior brings to mind the famous clock in the railway station that can only show discrete times. Note that by allowing l to increase, we may approximate a continuous clock as closely as we wish.

D. The uncertainty principle

Time operators, being ordinary operators, satisfy uncertainty relations with their canonical conjugates. Thus, for our linear clock, it follows from $[\theta,\eta]=i\hbar$ that θ and η satisfy the usual Heisenberg uncertainty relation $\Delta\theta\Delta\eta\geqslant\hbar/2$. The case of the continuous cyclic clock is mathematically more complicated (just as is the case of angle and angular momentum), and we will not discuss it here.[14] A very nice illustration of the uncertainty principle is provided by the discrete cyclic quantum clock. From Eq. (22) we see that in an eigenstate $|\phi_k\rangle$ of the time operator, all eigenstates of L appear with equal probability and the converse is also true. That is, if the value of ϕ_k is maximally certain, the value of L is maximally uncertain and conversely.

Note that in all these examples the conjugate operator of the time operator coincides with the Hamiltonian. However, this need not be generally the case; it is merely due to the simplicity of our examples (compare the case of a single particle where the conjugate momentum $\boldsymbol{p}$ coincides with the total momentum $\boldsymbol{P}$). If the quantum clock is part of a larger system, the conjugate of its time operator no longer coincides with the total Hamiltonian.[15]

V. A REMARK ON POSITION IN RELATIVISTIC QUANTUM MECHANICS

Because of the dominant role particles and rigid bodies play in classical physics, the notion of the position of a physical system seemed unproblematic. This is still true in nonrelativistic quantum mechanics, although we have seen that position operators may have discrete eigenvalues (and may not even exist for special systems). However, in relativistic quantum mechanics the concept of a position operator encounters serious problems. As could have been surmized from our remarks in Sec. II, a point particle can mimic the behavior of a point in space, but it cannot mimic the behavior of a point in spacetime. In a famous paper,[16] Newton and Wigner showed that the required behavior of a position operator under space translations and rotations almost uniquely determines this operator. However, the resulting operator $\boldsymbol{q}$ is noncovariant and, due to its energy being positive, has the ugly property that a state that is an eigenstate of it at a given time (a localized state) will be spread out over all of space an infinitesimal time later. This result has given rise to an extensive literature on the feasibility of a localizable particle concept in relativistic quantum theory.[17] For a Dirac spin-1/2 particle, the Newton–Wigner position operator turns out to be identical to the Foldy–Wouthuysen mean position operator.[18] From our point of view, this case is particularly interesting because when the Dirac equation was conceived in 1928, the space part $\boldsymbol{x}$ of the four-vector $x=(\boldsymbol{x},ct)$ appearing as the argument of Dirac's four-spinor wave function $\psi(x)$ was identified with the position of the electron. This identification had the embarrassing consequence that the corresponding velocity of the electron would always be found to be the velocity of light. It took twenty years before this problem was solved and the proper position operator $\boldsymbol{q}$ was identified. Here again, the notation $\boldsymbol{x}$ for both the particle position and the space coordinate certainly obscured the issue.

In relativistic quantum field theory neither the position of a particle nor the concept of a clock variable play a role. There, the basic quantity is the operator field $\phi(x)$, which is parametrized by the c-number coordinates $x=(\boldsymbol{x},ct)$ of spacetime points.

VI. CONCLUSION

When looking for a time operator in quantum mechanics, a distinction must be made between the universal time coordinate t, a c-number like the space coordinates, and dynamical time variables of physical systems situated *in* spacetime. Time variables stand in a particularly simple relation to t and do exist in specific physical systems: clocks. In the quantum formalism, position and time variables are not treated differently. Much of the confusion about time in quantum mechanics has been caused by not making a proper distinction, in classical as well as in quantum physics, between position variables of particles and coordinates of points of space. Dynamical position and time variables of material systems are essentially noncovariant quantities. The demands of relativistic covariance are so stringent in quantum mechanics that no concept of a point particle can meet them, rendering the discussion about dynamical time and position variables somewhat moot. The quantum field seems to be the more fruitful concept for incorporating relativistic covariance in quantum physics.

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[1]J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (1932). Translated by R. T. Beyer (Princeton University Press, Princeton, NJ, 1955), p. 354.
[2]A. Lahiri and P. B. Pal, *A First Book of Quantum Field Theory* (Alpha Science International Ltd., 2001), p. 2.
[3]A fine review is by P. Busch, "On the energy-time uncertainty relation (Parts I, II)," Found. Phys. **21**, 1–43 (1990). There is a considerable literature on classical notions of time, such as transit time and arrival time, in a quantum mechanical context. For a review see J. G. Muga and C. R. Leavens, "Arrival time in quantum mechanics," Phys. Rep. **338**, 353–438 (2000).
[4]The present article elaborates on ideas put forward in an earlier paper, J. Hilgevoord, "The uncertainty principle for energy and time," Am. J. Phys. **64**, 1451–1456 (1996).
[5]See, for example, H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, New York, 1980), Chap. 9.
[6]Reference 5, Sec. 9-5.
[7]One may object that the worldline of a particle can be given in a covariant form. But a worldline is not a particle: it has no energy and momentum.
[8]This model was suggested by P. Busch (private communication).
[9]The schizophrenic character of quantum mechanics in the treatment of space-time coordinates and dynamical variables was explicitly pointed out by Bohr: "The only significant point is that in each case some ultimate measuring instruments, like the scales and clocks which determine the frame of space-time coordination—on which, in the last resort, even the definitions of momentum and energy quantities rest—must always be described entirely on classical lines, and consequently kept outside the system subject to quantum mechanical treatment." See N. Bohr, The causality problem in modern physics, *New Theories in Physics* (International Institute of Intellectual Cooperation, Paris, 1939), pp. 11–45. This aspect of Bohr's doctrine has often been met with opposition, but it is a remarkable fact that it is firmly embedded in the *formalism* of standard quantum mechanics.

[10]In dealing with operators we shall adopt the level of mathematical precision of elementary texts on quantum mechanics. For our purpose it will suffice to point to analogies with well-known and generally accepted facts.

[11]The notation $\psi(\mathbf{x},t)$ for the wave function may also be responsible for the mysterious expression ''second quantization'' in quantum field theory where, in fact, the quantization of a field is meant.

[12]W. Pauli, *General Principles of Quantum Mechanics* (1933) (Springer-Verlag, New York, 1980), p. 63, footnote 2.

[13]A quantum clock of this type was discussed by E. P. Wigner, ''Relativistic Invariance and Quantum Phenomena'', Rev. Mod. Phys. **29**, 255–268 (1957). See also H. Salecker and E. P. Wigner, ''Quantum limitations of the measurement of space-time distances,'' Phys. Rev. **109**, 571–577 (1958); A. Peres, ''Measurement of time by quantum clocks,'' Am. J. Phys. **48**, 552–557 (1980); *Quantum Theory: Concepts and Methods* (Kluwer Academic, New York, 1993).

[14]J. Uffink, ''Measures of uncertainty and the uncertainty principle,'' University of Utrecht thesis, 1990.

[15]For a discussion of uncertainty relations of a different kind in which the total energy and momentum enter directly, see J. Hilgevoord, ''The uncertainty principle for energy and time II,'' Am. J. Phys. **66**, 396–402 (1998).

[16]T. D. Newton and E. P. Wigner, ''Localized states for elementary systems,'' Rev. Mod. Phys. **21**, 400–406 (1949).

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